

r -modes in Relativistic Superfluid Stars

Shijun Yoshida*

*Centro Multidisciplinar de Astrofísica – CENTRA, Departamento de Física,
Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal*

Umin Lee†

Astronomical Institute, Graduate School of Science, Tohoku University, Sendai 980-8578, Japan

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We discuss the modal properties of the r -modes of relativistic superfluid neutron stars, taking account of the entrainment effects between superfluids. In this paper, the neutron stars are assumed to be filled with neutron and proton superfluids and the strength of the entrainment effects between the superfluids are represented by a single parameter η . We find that the basic properties of the r -modes in a relativistic superfluid star are very similar to those found for a Newtonian superfluid star. The r -modes of a relativistic superfluid star are split into two families, ordinary fluid-like r -modes (r^o -mode) and superfluid-like r -modes (r^s -mode). The two superfluids counter-move for the r^s -modes, while they co-move for the r^o -modes. For the r^o -modes, the quantity $\kappa \equiv \sigma/\Omega + m$ is almost independent of the entrainment parameter η , where m and σ are the azimuthal wave number and the oscillation frequency observed by an inertial observer at spatial infinity, respectively. For the r^s -modes, on the other hand, κ almost linearly increases with increasing η . It is also found that the radiation driven instability due to the r^s -modes is much weaker than that of the r^o -modes because the matter current associated with the axial parity perturbations almost completely vanishes.

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I. INTRODUCTION

The r -mode instability [1, 2] belongs to the secular instability attributable to the so-called CFS mechanism [3, 4, 5] that drives unstable various modes of oscillation in a rotating star. The r -modes excited in neutron stars are expected to work as a mechanism of decelerating the spin velocity of neutron stars by emitting gravitational waves, which will be detectable by LIGO and other detectors [6, 7, 8, 9, 10]. For a recent review, see, e.g., [11].

In old and cold neutron stars having the interior temperature below the superfluid transition temperatures $T \sim 10^9$ K, neutrons in the inner crust and neutrons and protons in the core are believed to be in superfluid states [12]. It is therefore likely that most of observed neutron stars, which have cooled down very quickly via emission of neutrinos [13], have a core with superfluids. One of the important effects of rotating superfluids in cold neutron stars is mutual friction, which provides a strong dissipation mechanism for the oscillation modes [14]. The mutual friction is produced by scattering of normal fluid particles off the vortices in the rotating superfluids. Lindblom and Mendell [15] calculated the mutual friction damping associated with the r -modes in neutron stars with a superfluid core and concluded that the mutual friction in the core is ineffective to suppress the r -mode instability. Recently, Lee and Yoshida [16] have studied the r -modes in neutron stars with the superfluid core using a different numerical method, confirming most of the results obtained by [15]. Lee and Yoshida [16] have also shown that the r -modes are split into ordinary fluid-like r -modes and superfluid-like r -modes, as first suggested by Andersson and Comer [17], and that the instability caused by the superfluid-like r -modes is extremely weak and easily damped by dissipation processes in the interior. Very recently, Yoshida and Lee [18] have calculated inertial modes of superfluid neutron stars, and shown that the inertial modes are also split into two families, namely, ordinary fluid-like inertial modes and superfluid-like inertial modes. It is quite reasonable since the r -modes belong to a subclass of the inertial modes. Yoshida and Lee [18] found that all the inertial modes, except for the ordinary fluid-like r -modes, are strongly damped by the dissipation due to the mutual friction unless the entrainment effects between superfluids are extremely weak.

Most of the numerical studies on the r -modes and the inertial modes in superfluid neutron stars have been done within the framework of Newtonian dynamics. Since neutron stars are general relativistic objects having the typical relativistic factor $M(R)/R$ of order of 10^{-1} with $M(R)$ and R being the gravitational mass and radius of the star, respectively, it is necessary to extend the Newtonian analyses to general relativistic ones. In this paper, employing

*Electronic address: yoshida@fisica.ist.utl.pt

†Electronic address: lee@astr.tohoku.ac.jp

the two-constituent model developed by Carter and his co-workers [19, 20, 21, 22], and the formalism devised by Lockitch, Friedman, and Andersson [23, 24] for inertial modes of relativistic normal fluid stars, we derive a general relativistic formulation for inertial mode oscillations in superfluid neutron stars. In Sec. II we present the basic equations employed in this paper for the dynamics of relativistic superfluids, and in Sec. III numerical results are given, and Sec. IV is for discussions and conclusions. In this paper, we employ geometric units, given by $c = G = 1$, where c and G are the speed of light and the gravitational constant, respectively, and sign conventions used in [25].

II. FORMULATION

A. Two-constituent formalism for superfluid dynamics

We assume neutron stars are composed of superfluid neutrons and a mixture of superfluid protons and normal fluid electrons. We assume perfect charge neutrality between the protons and the electrons because the plasma frequency of the mixture is much higher than the oscillation frequency considered in this study [26]. The electrons therefore co-move with the protons. In other words, we do not need equations for describing the dynamics of the electrons. In the following, we loosely call the mixture of the protons and electrons “proton”. To describe the dynamics of the superfluids in a neutron star, we employ the relativistic two-constituent formalism, which has been developed by Carter and his co-workers [19, 20, 21, 22]. The fundamental quantity of Carter’s superfluid formalism is the so-called master function. Although we have several choices for the master function, we take, following Comer, Langlois, and Lin [22], the total thermodynamical energy density $-\Lambda$ as the master function. The master function Λ is assumed to depend on the three scalars $n^2 = -n_\alpha n^\alpha$, $p^2 = -p_\alpha p^\alpha$, and $x^2 = -p_\alpha n^\alpha$, where n^α and p^α are the conserved number density current of the neutrons and the protons, respectively.

Once an explicit functional form of the master function is given, the energy-momentum tensor T_β^α is given by

$$T_\beta^\alpha = \Psi \delta_\beta^\alpha + p^\alpha \chi_\beta + n^\alpha \mu_\beta, \quad (1)$$

where the scalar Ψ denotes the generalized pressure, defined by

$$\Psi = \Lambda - n^\alpha \mu_\alpha - p^\alpha \chi_\alpha, \quad (2)$$

and the one-forms μ_α and χ_α mean the chemical potential covectors, defined by

$$\mu_\alpha = \mathcal{B} n_\alpha + \mathcal{A} p_\alpha, \quad \chi_\alpha = \mathcal{A} n_\alpha + \mathcal{C} p_\alpha, \quad (3)$$

where

$$\mathcal{A} = -\frac{\partial \Lambda}{\partial x^2}, \quad \mathcal{B} = -2\frac{\partial \Lambda}{\partial n^2}, \quad \mathcal{C} = -2\frac{\partial \Lambda}{\partial p^2}. \quad (4)$$

The one-forms μ_α and χ_α are conjugate momenta to n^α and p^α , respectively. The quantities $\mu = (-\mu_\alpha \mu^\alpha)^{1/2}$ and $\chi = (-\chi_\alpha \chi^\alpha)^{1/2}$ are interpreted as the chemical potentials of the neutrons and the protons, respectively. From equation (3), we can see that the thermodynamical quantity \mathcal{A} determines the strength of the entrainment effects between the two superfluids. In other words, if we have $\mathcal{A} = 0$, there is no entrainment effect. The system of the dynamical equations for the two superfluids is composed of two continuity equations, given by

$$\nabla_\alpha n^\alpha = 0, \quad \nabla_\alpha p^\alpha = 0. \quad (5)$$

and two Euler equations, given by

$$n^\alpha \nabla_{[\alpha} \mu_{\beta]} = 0, \quad p^\alpha \nabla_{[\alpha} \chi_{\beta]} = 0, \quad (6)$$

where ∇_α means the covariant derivative associated with the metric tensor, and the square brackets denote anti-symmetrised averaging. Since n^α and p^α are the conserved currents, it is convenient to introduce two time-like unit vectors u^α and v^α defined by

$$n^\alpha = n u^\alpha, \quad p^\alpha = p v^\alpha, \quad (7)$$

where n and p denote the number density of the neutrons and the protons, respectively.

B. Equilibrium configurations

Equilibrium states of a slowly rotating star are described by the stationary axisymmetric spacetime, given by the line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - 2\omega r^2 \sin^2 \theta dt d\varphi. \quad (8)$$

Here, we have taken account of the rotational effects up to the first order of the stellar rotation angular frequency. In this study, we assume that the neutron and proton superfluids in equilibrium are in the same rotational motion with the uniform rotation angular frequency Ω . The four-velocity of the two superfluids is therefore given by

$$u^\alpha = v^\alpha = \gamma (t^\alpha + \Omega \varphi^\alpha) = e^{-\nu(r)} (t^\alpha + \Omega \varphi^\alpha), \quad (9)$$

where t^α and φ^α are the time-like and rotational Killing vectors of the spacetime. In virtue of equation (9), the conjugate momentum one-forms in equilibrium reduce to

$$\begin{aligned} \mu_\alpha &= (\mathcal{B}n + \mathcal{A}p) u_\alpha = \mu(n, p) u_\alpha, \\ \chi_\alpha &= (\mathcal{A}n + \mathcal{C}p) u_\alpha = \chi(n, p) u_\alpha, \end{aligned} \quad (10)$$

and the energy-momentum tensor of an equilibrium star is given by

$$T^\alpha_\beta = -\Lambda u^\alpha u_\beta + \Psi q^\alpha_\beta = -\Lambda u^\alpha u_\beta + (\Lambda + \mu n + \chi p) q^\alpha_\beta, \quad (11)$$

where q^α_β denotes the projection tensor associated with the four-velocity u^α , defined by $q^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta$.

Substituting equations (10) into Euler equations (6), we obtain the hydrostatic equations given by

$$\frac{1}{\mu} \frac{d\mu}{dr} = -\frac{d\nu}{dr}, \quad \frac{1}{\chi} \frac{d\chi}{dr} = -\frac{d\nu}{dr}, \quad (12)$$

which leads to

$$\frac{\mu}{\chi} = C, \quad (13)$$

where C is an integral constant. In this study, we take $C = 1$, assuming that the two superfluids are in chemical equilibrium in the equilibrium states. The metric coefficients are determined from the Einstein equations, which are written as

$$\frac{d\nu}{dr} = e^{2\lambda} \left(\frac{M}{r^2} + 4\pi r \Psi \right), \quad \frac{dM}{dr} = -4\pi r^2 \Lambda, \quad (14)$$

$$\frac{d^2 \varpi}{dr^2} + \left(\frac{4}{r} - \frac{d\nu}{dr} - \frac{d\lambda}{dr} \right) \frac{d\varpi}{dr} - \frac{4}{r} \left(\frac{d\nu}{dr} + \frac{d\lambda}{dr} \right) \varpi = 0, \quad (15)$$

where

$$M(r) = \frac{r}{2} (1 - e^{-2\lambda}), \quad \varpi = \Omega - \omega. \quad (16)$$

In this paper, we neglect a normal fluid envelope (e.g., [22]), and we require that the two superfluids have the common outer surface whose radius R is defined by equations $\mu(R) = m_n$ and $\chi(R) = m_p$, where m_n and m_p stand for the rest mass of the neutron and the proton, respectively. The surface boundary conditions for the equilibrium structure are given as follows:

$$\nu = \frac{1}{2} \ln \left(1 - \frac{2M(R)}{R} \right), \quad \varpi + \frac{R}{3} \frac{d\varpi}{dr} = \Omega, \quad \text{at } r = R, \quad (17)$$

where $M(R)$ stands for the gravitational mass of the star. The boundary conditions at the stellar center are the regularity condition for all the physical quantities.

C. Perturbation equations for the two-fluid model

To describe fluid perturbations in a star, it is convenient to introduce two kinds of changes in physical quantities, called the Eulerian and the Lagrangian changes [5]. An Eulerian change δQ is the difference between the quantities Q in the perturbed and the unperturbed states at a spacetime point. The relation between the Lagrangian changes and the Eulerian changes are given by

$$\Delta_n Q = \delta Q + L_{\xi_n} Q, \quad \Delta_p Q = \delta Q + L_{\xi_p} Q, \quad (18)$$

where L_k stands for the Lie derivative along the vector k^α , and we have introduced the two distinct Lagrangian displacement vectors ξ_n^α and ξ_p^α to describe the perturbed motion of the neutron and proton superfluids because the two superfluids can move independently. The Lagrangian displacement is regarded as a vector that connects fluid elements in the unperturbed state to the corresponding elements in the perturbed state.

We suppose that perturbations of any physical quantity can be expressed in terms of the Lagrangian displacements ξ_n^α and ξ_p^α , and the Eulerian changes in the metric $h_{\alpha\beta} = \delta g_{\alpha\beta}$. According to similar consideration to that in [5], we can write the Lagrangian changes of the fluid velocities of the neutron and proton in terms of the Lagrangian change in the metric:

$$\Delta_n u^\alpha = -\frac{1}{2} u^\alpha u^\mu u^\nu \Delta_n g_{\mu\nu}, \quad \Delta_p v^\alpha = -\frac{1}{2} u^\alpha u^\mu u^\nu \Delta_p g_{\mu\nu}, \quad (19)$$

where

$$\Delta_n g_{\alpha\beta} = h_{\alpha\beta} + \nabla_\alpha \xi_{n\beta} + \nabla_\beta \xi_{n\alpha}, \quad \Delta_p g_{\alpha\beta} = h_{\alpha\beta} + \nabla_\alpha \xi_{p\beta} + \nabla_\beta \xi_{p\alpha}. \quad (20)$$

Here, we have used the relation $u^\alpha = v^\alpha$ in the equilibrium. In this study, we consider no transfusion between the neutrons and the protons. The particle numbers of the neutrons and the protons are therefore conserved separately. Then, the conservation equations are given by

$$\frac{\Delta_n n}{n} = -\frac{1}{2} q^{\alpha\beta} \Delta_n g_{\alpha\beta}, \quad \frac{\Delta_p p}{p} = -\frac{1}{2} q^{\alpha\beta} \Delta_p g_{\alpha\beta}. \quad (21)$$

The Eulerian perturbations of the velocities and the number densities can be expressed in terms of ξ_n^α , ξ_p^α , and $h_{\alpha\beta}$ as

$$\begin{aligned} \delta u^\alpha &= q_\beta^\alpha L_u \xi_n^\beta + \frac{1}{2} u^\alpha u^\mu u^\nu h_{\mu\nu} = \delta \hat{u}^\alpha + \frac{1}{2} u^\alpha u^\mu u^\nu h_{\mu\nu}, \\ \delta v^\alpha &= q_\beta^\alpha L_u \xi_p^\beta + \frac{1}{2} u^\alpha u^\mu u^\nu h_{\mu\nu} = \delta \hat{v}^\alpha + \frac{1}{2} u^\alpha u^\mu u^\nu h_{\mu\nu}, \end{aligned} \quad (22)$$

$$\delta n = -\frac{n}{2} q^{\alpha\beta} h_{\alpha\beta} - q_\beta^\alpha \nabla_\alpha (n \xi_n^\beta), \quad \delta p = -\frac{p}{2} q^{\alpha\beta} h_{\alpha\beta} - q_\beta^\alpha \nabla_\alpha (p \xi_p^\beta). \quad (23)$$

According to [19] and assuming $u^\alpha = v^\alpha$ in the equilibrium, we can write the Eulerian perturbations of the conjugate momenta μ_α and χ_α as

$$\begin{aligned} \delta \mu_\alpha &= \mu \left(\delta \hat{u}_\alpha + h_{\alpha\beta} u^\beta + \frac{1}{2} u_\alpha u^\mu u^\nu h_{\mu\nu} \right) + p \mathcal{A} (\delta \hat{v}_\alpha - \delta \hat{u}_\alpha) + \{ (\mathcal{A} - \bar{\mathcal{A}}) \delta p + (\mathcal{B} - \bar{\mathcal{B}}) \delta n \} u_\alpha, \\ \delta \chi_\alpha &= \chi \left(\delta \hat{v}_\alpha + h_{\alpha\beta} u^\beta + \frac{1}{2} u_\alpha u^\mu u^\nu h_{\mu\nu} \right) + n \mathcal{A} (\delta \hat{u}_\alpha - \delta \hat{v}_\alpha) + \{ (\mathcal{C} - \bar{\mathcal{C}}) \delta p + (\mathcal{A} - \bar{\mathcal{A}}) \delta n \} u_\alpha, \end{aligned} \quad (24)$$

where

$$\begin{aligned} \bar{\mathcal{A}} &= -2np \frac{\partial \mathcal{B}}{\partial p^2} - 2n^2 \frac{\partial \mathcal{A}}{\partial n^2} - 2p^2 \frac{\partial \mathcal{A}}{\partial p^2} - np \frac{\partial \mathcal{A}}{\partial x^2}, \\ \bar{\mathcal{B}} &= -2n^2 \frac{\partial \mathcal{B}}{\partial n^2} - 4np \frac{\partial \mathcal{A}}{\partial n^2} - p^2 \frac{\partial \mathcal{A}}{\partial x^2}, \\ \bar{\mathcal{C}} &= -2p^2 \frac{\partial \mathcal{C}}{\partial p^2} - 4np \frac{\partial \mathcal{A}}{\partial p^2} - n^2 \frac{\partial \mathcal{A}}{\partial x^2}. \end{aligned} \quad (25)$$

Note that the second terms on the right-hand side of equations (24) represent the non-dissipative drag force between the two superfluids, which is proportional to both the function \mathcal{A} and the velocity difference $\delta\hat{v}_\alpha - \delta\hat{u}_\alpha$. From equation (6), the perturbed Euler equations are given by

$$\begin{aligned} q_\alpha^\beta L_u \delta\mu_\beta - q_\alpha^\beta \partial_\beta (u^\rho \delta\mu_\rho) + \mu \delta\hat{u}^\beta (\partial_\beta u_\alpha - \partial_\alpha u_\beta) &= 0, \\ q_\alpha^\beta L_u \delta\chi_\beta - q_\alpha^\beta \partial_\beta (u^\rho \delta\chi_\rho) + \chi \delta\hat{v}^\beta (\partial_\beta u_\alpha - \partial_\alpha u_\beta) &= 0, \end{aligned} \quad (26)$$

where ∂_α means the partial derivative.

It is convenient to introduce the vorticity equation when pulsations in a rotating star are considered. Making use of the conjugate momenta as dynamical variables, the vorticity equations can be obtained as

$$L_u d\mu_{\alpha\beta} = 0, \quad L_v d\chi_{\alpha\beta} = 0, \quad (27)$$

where $d\mu_{\alpha\beta}$ and $d\chi_{\alpha\beta}$ are the exterior differentiation of the one-forms μ_β and χ_β , defined by

$$d\mu_{\alpha\beta} = \partial_\alpha \mu_\beta - \partial_\beta \mu_\alpha, \quad d\chi_{\alpha\beta} = \partial_\alpha \chi_\beta - \partial_\beta \chi_\alpha. \quad (28)$$

Then, the Lagrangian variation of these equations can be derived straightforwardly:

$$\Delta_n L_u d\mu_{\alpha\beta} = L_u \Delta_n d\mu_{\alpha\beta} = 0, \quad \Delta_p L_v d\chi_{\alpha\beta} = L_v \Delta_p d\chi_{\alpha\beta} = 0, \quad (29)$$

where $\Delta_n d\mu_{\alpha\beta}$ and $\Delta_p d\chi_{\alpha\beta}$ can be written in terms of the Lagrangian changes in the conjugate momenta as

$$\Delta_n d\mu_{\alpha\beta} = d\Delta_n \mu_{\alpha\beta} = \partial_\alpha \Delta_n \mu_\beta - \partial_\beta \Delta_n \mu_\alpha, \quad \Delta_p d\chi_{\alpha\beta} = d\Delta_p \chi_{\alpha\beta} = \partial_\alpha \Delta_p \chi_\beta - \partial_\beta \Delta_p \chi_\alpha. \quad (30)$$

Note that since the equations $u^\beta d\Delta_n \mu_{\alpha\beta} = 0$ and $\partial_{[\alpha} d\Delta_n \mu_{\beta\gamma]} = 0$ ($v^\beta d\Delta_p \chi_{\alpha\beta} = 0$ and $\partial_{[\alpha} d\Delta_p \chi_{\beta\gamma]} = 0$) are satisfied, $d\Delta_n \mu_{\alpha\beta}$ ($d\Delta_p \chi_{\alpha\beta}$) has only two independent components.

The metric perturbations $h_{\alpha\beta}$ are determined by the linearized Einstein equations, given by

$$\delta G_\beta^\alpha = 8\pi \delta T_\beta^\alpha, \quad (31)$$

where δG_β^α denotes the linearized Einstein tensor. Here, δT_β^α means the linearized energy-momentum tensor, and is given by

$$\delta T_\beta^\alpha = \delta\Psi \delta_\beta^\alpha + \delta n^\alpha \mu_\beta + \delta p^\alpha \chi_\beta + n^\alpha \delta\mu_\beta + p^\alpha \delta\chi_\beta, \quad (32)$$

where

$$\delta\Psi = \frac{1}{2}(n^\alpha \mu^\beta + p^\alpha \chi^\beta) h_{\alpha\beta} - n^\alpha \delta\mu_\alpha - p^\alpha \delta\chi_\alpha. \quad (33)$$

D. Equations of state: Expanded master function

In this study, we make use of the same master function as that introduced in [27], which is generally given by

$$\Lambda(n^2, p^2, x^2) = \sum_{i=0}^{\infty} \lambda_i(n^2, p^2) (x^2 - np)^i. \quad (34)$$

This expansion of the master function may be justified because $x^2 - np = 0$ in equilibrium and the deviation from $x^2 - np = 0$ for perturbed states is the same order of the perturbations. Although Comer and Joynt [28] have recently discussed in greater detail the entrainment effects in general relativistic superfluids, we use the master function given above for simplicity. In terms of the master function (34), we can obtain the thermodynamical functions in pulsation equations as follows:

$$\begin{aligned} \mathcal{A} &= -\lambda_1(n^2, p^2), \quad \mathcal{B} = -\frac{p}{n}\mathcal{A} - \frac{1}{n}\frac{\partial\lambda_0}{\partial n}, \quad \mathcal{C} = -\frac{n}{p}\mathcal{A} - \frac{1}{p}\frac{\partial\lambda_0}{\partial p}, \\ \bar{\mathcal{A}} &= \mathcal{A} + \frac{\partial^2\lambda_0}{\partial n\partial p}, \quad \bar{\mathcal{B}} = \mathcal{B} + \frac{\partial^2\lambda_0}{\partial n^2}, \quad \bar{\mathcal{C}} = \mathcal{C} + \frac{\partial^2\lambda_0}{\partial p^2}, \end{aligned} \quad (35)$$

where we have used the relation $x^2 = np$ in the equilibrium. Note that $\bar{\mathcal{A}}$ becomes equal to \mathcal{A} if all the expansion coefficients λ_i are separable in n and p in the sense that $\lambda_i = f_i(n) + g_i(p)$ for appropriate functions $f_i(n)$ and $g_i(p)$. Using the chemical potentials in the equilibrium given by

$$\mu = -\frac{\partial\lambda_0}{\partial n}, \quad \chi = -\frac{\partial\lambda_0}{\partial p}, \quad (36)$$

we have

$$\mathcal{A} - \bar{\mathcal{A}} = \frac{\partial\mu}{\partial p} = \frac{\partial\chi}{\partial n}, \quad \mathcal{B} - \bar{\mathcal{B}} = \frac{\partial\mu}{\partial n}, \quad \mathcal{C} - \bar{\mathcal{C}} = \frac{\partial\chi}{\partial p}, \quad (37)$$

and, assuming all the expansion coefficients λ_i are separable in n and p , we obtain

$$\mathcal{A} = \bar{\mathcal{A}}, \quad \mathcal{B} - \bar{\mathcal{B}} = \frac{d\mu}{dn}, \quad \mathcal{C} - \bar{\mathcal{C}} = \frac{d\chi}{dp}. \quad (38)$$

E. Oscillation equations for inertial modes

If we assume the star in the equilibrium is stationary and axisymmetric, the time and azimuthal dependence of the perturbations can be given by $\exp(i\sigma t + im\varphi)$, where σ is the oscillation frequency observed by an inertial observer at spatial infinity, and m is the azimuthal wave number. Because of the rotation effects, it is generally impossible to achieve the separation of variables for the perturbed quantities. We expand the perturbations in terms of the tensor spherical harmonics with different l 's for a given m . In this paper, we consider the oscillation modes associated with $m \geq 2$ because we are interested in the modes that are unstable against gravitational radiation reactions. Thus, we can select the so-called Regge-Wheeler gauge in order to fix the gauge freedom for the metric perturbations [29]. Then, the metric perturbations are expanded as

$$\begin{aligned} h_{\mu\nu} = & \sum_{l \geq |m|} \begin{pmatrix} e^{2\nu} H_{0l}(r) & H_{1l}(r) & 0 & 0 \\ \text{sym} & e^{2\lambda} H_{2l}(r) & 0 & 0 \\ \text{sym} & \text{sym} & K_l(r)r^2 & 0 \\ \text{sym} & \text{sym} & \text{sym} & r^2 \sin^2 \theta K_l(r) \end{pmatrix} Y_l^m(\theta, \varphi) e^{i\sigma t}, \\ & + \sum_{l' \geq |m|} \begin{pmatrix} 0 & 0 & i h_{0l'}(r) \frac{-1}{\sin \theta} \partial_\varphi Y_{l'}^m(\theta, \varphi) & i h_{0l'}(r) \sin \theta \partial_\theta Y_{l'}^m(\theta, \varphi) \\ \text{sym} & 0 & i h_{1l'}(r) \frac{-1}{\sin \theta} \partial_\varphi Y_{l'}^m(\theta, \varphi) & i h_{1l'}(r) \sin \theta \partial_\theta Y_{l'}^m(\theta, \varphi) \\ \text{sym} & \text{sym} & 0 & 0 \\ \text{sym} & \text{sym} & \text{sym} & 0 \end{pmatrix} e^{i\sigma t}, \end{aligned} \quad (39)$$

where $Y_l^m(\theta, \varphi)$ is the spherical harmonic function, and $l = |m| + 2(k-1)$ and $l' = l+1$ for even modes, and $l = |m| + 2k-1$ and $l' = l-1$ for odd modes, where $k = 1, 2, 3, \dots$. The even and odd modes are characterized by symmetry and antisymmetry of the eigenfunctions with respect to the equatorial plane. The Lagrangian displacements $\xi_{n\alpha}$ and $\xi_{p\alpha}$ can be expanded in terms of the vector spherical harmonics as

$$i\kappa\Omega\xi_{nt} = 0, \quad i\kappa\Omega\xi_{pt} = 0, \quad (40)$$

$$i\kappa\Omega\xi_{nr} = \sum_{l \geq |m|} \frac{e^{2\lambda}}{r} W_{nl}(r) Y_l^m(\theta, \varphi) e^{i\sigma t}, \quad i\kappa\Omega\xi_{pr} = \sum_{l \geq |m|} \frac{e^{2\lambda}}{r} W_{pl}(r) Y_l^m(\theta, \varphi) e^{i\sigma t}, \quad (41)$$

$$\begin{aligned} i\kappa\Omega\xi_{n\theta} &= \sum_{l, l' \geq |m|} \left(V_{nl}(r) \partial_\theta Y_l^m(\theta, \varphi) - i U_{nl'} \frac{1}{\sin \theta} \partial_\varphi Y_{l'}^m(\theta, \varphi) \right) e^{i\sigma t}, \\ i\kappa\Omega\xi_{p\theta} &= \sum_{l, l' \geq |m|} \left(V_{pl}(r) \partial_\theta Y_l^m(\theta, \varphi) - i U_{pl'} \frac{1}{\sin \theta} \partial_\varphi Y_{l'}^m(\theta, \varphi) \right) e^{i\sigma t}, \end{aligned} \quad (42)$$

$$\begin{aligned} i\kappa\Omega\xi_{n\varphi} &= \sum_{l, l' \geq |m|} (V_{nl}(r) \partial_\varphi Y_l^m(\theta, \varphi) + i U_{nl'} \sin \theta \partial_\theta Y_{l'}^m(\theta, \varphi)) e^{i\sigma t}, \\ i\kappa\Omega\xi_{p\varphi} &= \sum_{l, l' \geq |m|} (V_{pl}(r) \partial_\varphi Y_l^m(\theta, \varphi) + i U_{pl'} \sin \theta \partial_\theta Y_{l'}^m(\theta, \varphi)) e^{i\sigma t}, \end{aligned} \quad (43)$$

where $\kappa\Omega \equiv \sigma + m\Omega$ is the oscillation frequency in the corotating frame of the star. Note that the gauge freedom for the Lagrangian displacement is fixed so as to satisfy $\xi_{nt} = \xi_{pt} = 0$. The Eulerian perturbations of the number densities for the neutrons and protons are given by

$$\delta n = \sum_{l \geq |m|} \delta n_l(r) Y_l^m(\theta, \varphi) e^{i\sigma t}, \quad \delta p = \sum_{l \geq |m|} \delta p_l(r) Y_l^m(\theta, \varphi) e^{i\sigma t}. \quad (44)$$

When we are interested in inertial modes in a slowly rotating superfluid star in the lowest order in the rotation frequency Ω , it is rather easy to generalize the formulation devised by Lockitch et al. [23] for relativistic inertial modes in normal fluid neutron stars. According to [23], we assume that the perturbations satisfy the following ordering laws in the slow rotation limit of $\Omega \rightarrow 0$:

$$\begin{aligned} H_1 &= O(1), & W_n &= O(1), & W_p &= O(1), & V_n &= O(1), & V_p &= O(1), \\ H_0 &= O(\Omega), & H_2 &= O(\Omega), & K &= O(\Omega), & \delta n &= O(\Omega), & \delta p &= O(\Omega), \end{aligned} \quad (45)$$

$$h_0 = O(1), \quad U_p = O(1), \quad U_n = O(1), \quad (46)$$

and $\kappa = O(1)$. Comer [30] showed that solutions obeying these ordering laws are allowed in the perturbation equations for rotating stars, exploring the so-called zero-frequency subspace of eigensolutions to perturbation equations in a non-rotating relativistic superfluid star. If we consider no rotational effects, the solutions subject to the ordering laws (45) are interpreted as infinitely degenerate g -modes of zero-frequency, while the solutions subject to the ordering laws (46) are considered to be a relativistic and superfluid counterpart of the trivial toroidal mode in a non-rotating Newtonian normal fluid star [31].

Substituting the perturbed quantities into linearized equations (23), (29), and (31) and assuming the ordering laws for the eigenfunctions (45) and (46), we can obtain a system of infinitely coupled ordinary differential equations for inertial modes in a slowly rotating superfluid star. To write down the oscillation equations for the inertial modes, it is convenient to use vector notation for the eigenfunctions \mathbf{w}_n , \mathbf{w}_p , \mathbf{v}_n , \mathbf{v}_p , \mathbf{u}_n , \mathbf{u}_p , \mathbf{h} , and \mathbf{H} , whose components are given by

$$w_{n,k} = W_{nl}, \quad w_{p,k} = W_{pl}, \quad v_{n,k} = V_{nl}, \quad v_{p,k} = V_{pl}, \quad u_{n,k} = U_{nl'}, \quad u_{p,k} = U_{pl'}, \quad h_k = h_{0l'}, \quad H_k = H_{1l}, \quad (47)$$

where $l = |m| + 2(k-1)$, $l' = l+1$ for even modes and $l = |m| + 2k-1$, $l' = l-1$ for odd modes, where $k = 1, 2, 3, \dots$. The particle number conservation equations (23) for the neutron and proton are then written as

$$\begin{aligned} e^{2\nu} r \frac{d(e^{-2\nu} \mathbf{w}_n)}{dr} &= - \left(1 + \frac{r}{n} \frac{dn}{dr} + r \frac{d\lambda}{dr} + 2r \frac{d\nu}{dr} \right) \mathbf{w}_n + \Lambda_0 \mathbf{v}_n, \\ e^{2\nu} r \frac{d(e^{-2\nu} \mathbf{w}_p)}{dr} &= - \left(1 + \frac{r}{p} \frac{dp}{dr} + r \frac{d\lambda}{dr} + 2r \frac{d\nu}{dr} \right) \mathbf{w}_p + \Lambda_0 \mathbf{v}_p. \end{aligned} \quad (48)$$

Independent components of the vorticity equations (29) lead to

$$\begin{aligned} L_1 \mathbf{u}_n - q M_0 \mathbf{v}_n + \bar{q} K_0 \mathbf{w}_n + \mathcal{A}_1 (\mathbf{u}_p - \mathbf{u}_n) + \mathbf{h} &= 0, \\ L_1 \mathbf{u}_p - q M_0 \mathbf{v}_p + \bar{q} K_0 \mathbf{w}_p + \mathcal{A}_2 (\mathbf{u}_n - \mathbf{u}_p) + \mathbf{h} &= 0, \end{aligned} \quad (49)$$

$$\begin{aligned} L_0 e^{2\nu} r \frac{d(e^{-2\nu} \mathbf{v}_n)}{dr} &- q M_1 e^{2\nu} r \frac{d(e^{-2\nu} \mathbf{u}_n)}{dr} - m \bar{q} \Lambda_0^{-1} e^{2\nu} r \frac{d(e^{-2\nu} \mathbf{w}_n)}{dr} - \left(e^{2\lambda} + m r \frac{d\bar{q}}{dr} \Lambda_0^{-1} \right) \mathbf{w}_n \\ &+ \left(\bar{q} K_1 \Lambda_1 - r \frac{dq}{dr} M_1 \right) \mathbf{u}_n + m \left(\bar{q} - r \frac{dq}{dr} \Lambda_0^{-1} \right) \mathbf{v}_n - r \mathbf{H} \\ &+ r \frac{d\mathcal{A}_1}{dr} (\mathbf{v}_p - \mathbf{v}_n) + \mathcal{A}_1 \left(e^{2\lambda} (\mathbf{w}_n - \mathbf{w}_p) + e^{2\nu} r \frac{d}{dr} (e^{-2\nu} \mathbf{v}_p - e^{-2\nu} \mathbf{v}_n) \right) = 0, \\ L_0 e^{2\nu} r \frac{d(e^{-2\nu} \mathbf{v}_p)}{dr} &- q M_1 e^{2\nu} r \frac{d(e^{-2\nu} \mathbf{u}_p)}{dr} - m \bar{q} \Lambda_0^{-1} e^{2\nu} r \frac{d(e^{-2\nu} \mathbf{w}_p)}{dr} - \left(e^{2\lambda} + m r \frac{d\bar{q}}{dr} \Lambda_0^{-1} \right) \mathbf{w}_p \\ &+ \left(\bar{q} K_1 \Lambda_1 - r \frac{dq}{dr} M_1 \right) \mathbf{u}_p + m \left(\bar{q} - r \frac{dq}{dr} \Lambda_0^{-1} \right) \mathbf{v}_p - r \mathbf{H} \\ &+ r \frac{d\mathcal{A}_2}{dr} (\mathbf{v}_n - \mathbf{v}_p) + \mathcal{A}_2 \left(e^{2\lambda} (\mathbf{w}_p - \mathbf{w}_n) + e^{2\nu} r \frac{d}{dr} (e^{-2\nu} \mathbf{v}_n - e^{-2\nu} \mathbf{v}_p) \right) = 0, \end{aligned} \quad (50)$$

where

$$q = 2\kappa^{-1}\frac{\varpi}{\Omega}, \quad \bar{q} = \kappa^{-1}\frac{e^{2\nu}}{r}\frac{d}{dr}\left(e^{-2\nu}r^2\frac{\varpi}{\Omega}\right), \quad \mathcal{A}_1 = \frac{p}{\mu}\mathcal{A}, \quad \mathcal{A}_2 = \frac{n}{\chi}\mathcal{A}. \quad (51)$$

The linearized Einstein equations are reduced to

$$r\mathbf{H} = -\Lambda_0^{-1}16\pi e^{2\lambda}r^2(n\mu\mathbf{w}_n + p\chi\mathbf{w}_p), \quad (52)$$

$$r\frac{d}{dr}\left(r\frac{d}{dr}\mathbf{h}\right) - \left(1 + r\frac{d\nu}{dr} + r\frac{d\lambda}{dr}\right)r\frac{d}{dr}\mathbf{h} - \left(e^{2\lambda}\Lambda_1 + 2\left(1 - e^{2\lambda} + r\frac{d\nu}{dr} + r\frac{d\lambda}{dr}\right)\mathbf{1}\right)\mathbf{h} = 16\pi e^{2\lambda}r^2(n\mu\mathbf{u}_n + p\chi\mathbf{u}_p) \quad (53)$$

where $\mathbf{1}$ stands for the unit matrix. Here, the matrices $L_0, L_1, K_0, K_1, \Lambda_0, \Lambda_1, M_0$, and M_1 are defined in Appendix. The oscillation equations we use are given as a set of linear ordinary differential equations for the variables $\mathbf{w}_n, \mathbf{w}_p, \mathbf{u}_n, \mathbf{u}_p, \mathbf{h}$, and $r\frac{d\mathbf{h}}{dr}$, which are obtained by eliminating the vectors $\mathbf{v}_n, \mathbf{v}_p$, and \mathbf{H} in equations (48), (50), and (53) using the algebraic equations (49) and (52).

The boundary conditions imposed at the center of the star are regularity condition that all the perturbation functions are regular and do not diverge at $r = 0$. This implies that the eigenfunctions we solve must vanish at $r = 0$. In order to determine the boundary conditions at the stellar surface, on the other hand, we follow the arguments similar to those given in [22]. In our specific equilibrium models, for which we have assumed an equation of state that is separable in n and p , the generalized pressure Ψ of the fluids can be written as $\Psi(n, p) = \Psi_n(n) + \Psi_p(p)$, where $\Psi_n(n)$ and $\Psi_p(p)$ represent the pressures due to the neutron and proton superfluids, respectively. Since the two superfluids can flow independently, as appropriate surface boundary conditions we require that the Lagrangian changes in each of the pressures vanish at the free surface of the star, which leads to

$$\Delta_n\Psi_n = \frac{d\Psi_n}{dn}\Delta_n n = 0, \quad \Delta_p\Psi_p = \frac{d\Psi_p}{dp}\Delta_n p = 0. \quad (54)$$

The conditions $\Delta_n n = 0$ and $\Delta_p p = 0$ at the surface therefore result in the surface boundary conditions given by $W_{nl} = 0$ and $W_{pl} = 0$ for inertial mode solutions in the lowest order of Ω . In the exterior of the star, the only non-trivial metric perturbations $h_{0l'}$ are determined by equation (53), which has two independent solutions, one is regular at spatial infinity, the other singular. The regular solution can be given as a series expansion [23, 24]:

$$h_{0l'} = \sum_{s=0}^{\infty} \hat{h}_{l',s} \left(\frac{R}{r}\right)^{l'+s}, \quad (55)$$

where

$$\hat{h}_{l',s} = \frac{(l' + s - 2)!(l' + s + 1)!(2l' + 1)!}{s!(l' - 2)!(l' + 1)!(2l' + s + 1)!} \left(\frac{2M(R)}{R}\right)^s \hat{h}_{l',0}, \quad (56)$$

and $\hat{h}_{l',0}$ is an arbitrary constant. If we require that the interior solution $h_{0l'}$ must be continuous with the exterior solution (55) at the stellar surface because the spacetime must be regular everywhere, we obtain the boundary conditions for the metric perturbations at the stellar surface given by

$$\lim_{\epsilon \rightarrow 0} \left[h_{0l'}(R + \epsilon) r \frac{d}{dr} h_{0l'}(R - \epsilon) - r \frac{d}{dr} h_{0l'}(R + \epsilon) h_{0l'}(R - \epsilon) \right] = 0. \quad (57)$$

For numerical computation, oscillation equations of a finite dimension are obtained by disregarding the terms with l larger than l_{max} in the expansions of the perturbations, where l_{max} is determined so that the eigenfrequency and the eigenfunctions are well converged as l_{max} increases. We solve the oscillation equations of a finite dimension as an eigenvalue problem with the scaled oscillation frequency κ using a Henyey type relaxation method (see, e.g., [31]).

III. NUMERICAL RESULTS

A. Equilibrium models

For the equilibrium structure with $x^2 - np = 0$, we use a generalized polytropic equation of state, introduced by Comer et al. [22], which is given by

$$\lambda_0(n^2, p^2) = -m_n n - \sigma_n n^{\beta_n} - m_p p - \sigma_p p^{\beta_p}, \quad (58)$$

where σ_n , σ_p , β_n , and β_p are constants. Note that this master function is separable in n and p . The generalized pressure and chemical potentials in the equilibrium can then be written as

$$\begin{aligned}\Psi &= \sigma_n(\beta_n - 1)n^{\beta_n} + \sigma_p(\beta_p - 1)p^{\beta_p}, \\ \mu &= m_n + \sigma_n\beta_n n^{\beta_n-1}, \\ \chi &= m_p + \sigma_p\beta_p p^{\beta_p-1}.\end{aligned}\tag{59}$$

Because chemical equilibrium is assumed in the equilibrium state, we can analytically write p in terms of n as

$$p = \left(\frac{\sigma_n\beta_n}{\sigma_p\beta_p}\right)^{1/(\beta_p-1)} n^{(\beta_n-1)/(\beta_p-1)},\tag{60}$$

where we have assumed $m_p = m_n$. We confirm from equations (59) and (60) that two superfluids in an equilibrium state given by equation (58) have the common outer surface if β_n and β_p satisfy the conditions $\beta_n \geq 1$ and $\beta_p \geq 1$. In order to introduce the entrainment effects in the perturbed states with $x^2 - np \neq 0$, we include, according to [27], the expansion coefficient λ_1 , given by

$$\lambda_1 = -\eta \frac{m_n m_p}{m_p p + \eta(m_n n + m_p p)},\tag{61}$$

where η is a parameter whose appropriate range is considered to be $0.04 \leq \eta \leq 0.2$ [27, 32, 33]. In this paper, we call η the entrainment parameter. Making use of the formulas obtained in the preceding sections, we can explicitly write the relevant thermodynamical coefficients as follows:

$$\mathcal{A} = \bar{\mathcal{A}} = \eta \frac{m_n m_p}{m_p p + \eta(m_n n + m_p p)}.\tag{62}$$

In order to avoid unnecessary singular behaviors of p and n at the stellar surface, we consider only the case of $\beta_n = \beta_p$. Thus, all the models we use are non-stratified models in the sense that $n/p = \text{constant}$ inside the star. The constant parameters σ_n , σ_p , β_n , and β_p in λ_0 for the equilibrium equation of state (see Table 1) are the same as those used for model 1 calculated in [22]. In this paper, we consider two equilibrium models, whose physical parameters are tabulated in Table I. The model I is almost Newtonian whose relativistic factor $M(R)/R$ is $M(R)/R = 10^{-3}$, while the model II is a relativistic one with the relativistic factor $M(R)/R = 0.15$, the value of which is similar to those for the models used in [24]. Although the model I is not appropriate as a physical model of neutron stars, we use it to compare with completely Newtonian calculations [16, 18].

B. r -mode oscillations

We calculated the r -modes of the two equilibrium models for a wide range of the entrainment parameter. It is important to note that the $l' = m$ fundamental r -modes, whose eigenfunction U_{nm} and/or U_{pm} is dominating and has no nodes in radial direction, are the only r -modes we find in this study. This situation is quite similar to that for the r -modes in a barotropic ordinary fluid star, in which neither the overtone r -modes with $l' = m$ nor the r -modes with $l' \neq m$ are found [24, 34, 35, 36]. It is generally found that the r -modes in a relativistic superfluid star are split into two families, which we call ordinary fluid r -modes (r^o -modes) and superfluid r -modes (r^s -modes), according to the notation of Lee and Yoshida [16]. Because of doubling of the dynamical degrees of freedom for the system of two superfluids, the mode splitting of this kind also appears in other oscillation modes essentially associated with fluid motions [16, 17, 18, 27, 37, 38, 39, 40]. In Figures 1 and 2, we plot the scaled frequencies $\kappa \equiv \sigma/\Omega + m$ of the r -modes associated with $m = 2$ and 3 as a function of the entrainment parameter η . As shown by the figures, although κ of the r^s -modes increases almost linearly with increasing η , κ of the r^o -modes is almost independent of η . Comparing the two frequency curves for the models I and II, it is found that both the r^o -mode and r^s -mode frequencies are strongly dependent on the relativistic factor $M(R)/R$ of the equilibrium models. This is because the frequency of the r -modes is roughly proportional to an *average* of the function ϖ within the star, which reflects the general relativistic effects such as the rotational frame dragging and the gravitational redshift. Similar dependence of κ on the parameter $M(R)/R$ for the r -modes has been found for relativistic ordinary fluid stars [23, 24, 41, 42, 43, 44].

From Table II, where κ 's for the r^o - and r^s -modes associated with $m = 2$ and 3 are tabulated for several values of η , we can confirm that κ 's for the r^o -modes for model I are nearly equal to $2m/l'(l' + 1)$, the value found for the r -modes in Newtonian ordinary fluid stars. It is important to note that κ 's for the r^o - and r^s -modes at $\eta = 0$ are not equal to each other, and the difference between the κ 's increases as the relativistic factor $M(R)/R$ is increased. In a Newtonian star, however, κ 's for the r^o - and r^s -modes at $\eta = 0$ are equal to each other in the lowest order of Ω , as

shown in [16]. The reason for the difference between the Newtonian and relativistic r -modes at $\eta = 0$ is that even at $\eta = 0$ there still exist couplings between two superfluid motions through the Einstein equations in the relativistic case, but no such couplings exist in the Newtonian case because all the Newtonian gravitational perturbations vanish in the lowest order of Ω .

In order to illustrate how the eigenfunctions of the r^o - and r^s -modes in a relativistic superfluid star behave, it is convenient to introduce a new set of variables defined by

$$\begin{aligned} W_{+l} &= \frac{n W_{nl} + p W_{pl}}{n + p}, & W_{-l} &= W_{nl} - W_{pl}, \\ V_{+l} &= \frac{n V_{nl} + p V_{pl}}{n + p}, & V_{-l} &= V_{nl} - V_{pl}, \\ U_{+l'} &= \frac{n U_{nl'} + p U_{pl'}}{n + p}, & U_{-l'} &= U_{nl'} - U_{pl'}. \end{aligned} \quad (63)$$

In Figures 3 and 4, we display four dominant coefficients U_{+2} , W_{+3} , U_{+4} , and h_{02} for the r^o -mode, and U_{-2} , W_{-3} , U_{-4} , and h_{02} for the r^s -mode for the case of $m = 2$ and $\eta = 0.1$ for model II. Here, the normalization condition has been given by $U_{nm} = 1$ at $r = R$. The coefficients that are not displayed in these figures have very small and negligible amplitudes. Figures 3 and 4 represent that the basic properties of the eigenfunctions are quite similar to those of the r^o - and r^s -modes in a Newtonian star with superfluidity [16, 18]. The neutrons and protons co-move for the r^o -modes, while they counter-move for the r^s -modes. It is also found in Figure 4 that the metric perturbation h_{02} almost completely vanishes for the r^s -modes, which is consistent with the fact that $U_{+m} \sim 0$. This means that gravitational radiations due to r^s -mode oscillations are negligible at least in the lowest order in Ω . It is important to note that the other coefficients $W_{\pm 3}$ and $U_{\pm 4}$ are not necessarily negligible compared with $U_{\pm 2}$ because of the general relativistic effects [23, 24, 44].

IV. DISCUSSIONS AND CONCLUSIONS

In this paper we have studied the modal properties of the r -modes of relativistic superfluid neutron stars, taking account of the entrainment effects between the neutron and proton superfluids. To describe the general relativistic dynamics of the superfluids, we employed the two-constituent formalism developed by Carter and his co-workers [19, 20, 21, 22]. We derived the perturbation equations for the relativistic inertial modes in neutron stars filled with the superfluids by generalizing the formalism employed by Lockitch, Andersson, and Friedman [23, 24] for the relativistic inertial modes in normal fluid neutron stars. We found that the basic properties of the r -modes in a relativistic star with the two superfluids are very similar to those in a Newtonian superfluid star [16, 18]. We confirmed that the r -modes of relativistic superfluid stars are split into two families, ordinary fluid-like r -modes (r^o -mode) and superfluid-like r -modes (r^s -mode). The two superfluids counter-move for the r^s -modes, while they co-move for the r^o -modes. The dimensionless frequency κ for the r^o -modes is almost independent of the entrainment parameter η . For the r^s -modes, on the other hand, κ almost linearly increases with η . The gravitational radiation driven instability due to the r^s -modes is much weaker than that of the r^o -modes because the matter current associated with the axial parity perturbations vanish almost completely for the former.

A solid crust near the surface of the neutron star has a significant influence on the modal properties of the r -modes in the superfluid core, since the solid crust supports its own oscillation modes [45, 46, 47, 48], and resonance phenomena are expected between the r -modes in the superfluid core and the torsional sound waves in the solid crust even in the general relativistic context [49, 50]. Dissipation in the viscous boundary layer at the interface between the fluid core and the solid crust is another important issue for the r -mode instability, as shown by [51]. How the dissipation in the core-crust interface can be important for the r -mode instability will be a quite interesting problem when superfluid neutrons and protons in the core and superfluid neutrons in the inner crust are taken into account simultaneously (see, e.g., [52]).

Newtonian superfluid neutron stars can support infinite number of inertial modes, which are also split into two families, ordinary fluid-like inertial modes (i^o -mode) and superfluid-like inertial modes (i^s -mode) [16, 18]. Non-linear couplings between the r -modes and inertial modes will be important to limit the amplitude growth of the r -mode instability, as recently shown by Arras et al. [53] for Newtonian normal fluid stars. To investigate the relativistic inertial modes in superfluid neutron stars will be one of our future studies.

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Appendix: Matrices used in pulsation equations

The components of the matrices, L_0 , L_1 , K_0 , K_1 , Λ_0 , Λ_1 , M_0 , and M_1 are given as follows:
For even modes,

$$(L_0)_{i,i} = 1 - \frac{mq}{l(l+1)}, \quad (L_1)_{i,i} = 1 - \frac{mq}{(l+1)(l+2)},$$

$$(K_0)_{i,i} = \frac{J_{l+1}^m}{l+1}, \quad (K_0)_{i,i+1} = -\frac{J_{l+2}^m}{l+2},$$

$$(K_1)_{i,i} = -\frac{J_{l+1}^m}{l+1}, \quad (K_1)_{i+1,i} = \frac{J_{l+2}^m}{l+2},$$

$$(\Lambda_0)_{i,i} = l(l+1), \quad (\Lambda_1)_{i,i} = (l+1)(l+2),$$

$$(M_0)_{i,i} = \frac{l}{l+1} J_{l+1}^m, \quad (M_0)_{i,i+1} = \frac{l+3}{l+2} J_{l+2}^m,$$

$$(M_1)_{i,i} = \frac{l+2}{l+1} J_{l+1}^m, \quad (M_1)_{i+1,i} = \frac{l+1}{l+2} J_{l+2}^m,$$

where $l = |m| + 2i - 2$ for $i = 1, 2, 3, \dots$.

For odd modes,

$$(L_0)_{i,i} = 1 - \frac{mq}{l(l+1)}, \quad (L_1)_{i,i} = 1 - \frac{mq}{l(l-1)},$$

$$(K_0)_{i,i} = -\frac{J_l^m}{l}, \quad (K_0)_{i+1,i} = \frac{J_{l+1}^m}{l+1},$$

$$(K_1)_{i,i} = \frac{J_l^m}{l}, \quad (K_1)_{i,i+1} = -\frac{J_{l+1}^m}{l+1},$$

$$(\Lambda_0)_{i,i} = l(l+1), \quad (\Lambda_1)_{i,i} = l(l-1),$$

$$(M_0)_{i,i} = \frac{l+1}{l} J_l^m, \quad (M_0)_{i+1,i} = \frac{l}{l+1} J_{l+1}^m,$$

$$(M_1)_{i,i} = \frac{l-1}{l} J_l^m, \quad (M_1)_{i,i+1} = \frac{l+2}{l+1} J_{l+1}^m,$$

where $l = |m| + 2i - 1$ for $i = 1, 2, 3, \dots$. Here, J_l^m is the function of m and l , defined by $J_l^m = [(l^2 - m^2)/(4l^2 - 1)]^{1/2}$, and $q = 2\varpi/\kappa\Omega$.

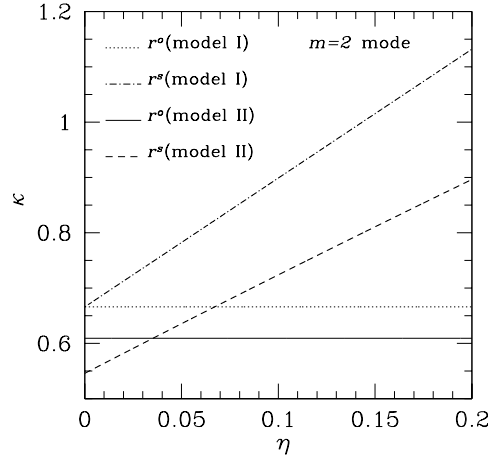
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TABLE I: Parameters describing stellar models I and II.

	model I	model II
σ_n/m_n	0.2	0.2
σ_p/m_n	0.5	0.5
β_n	2.0	2.0
β_p	2.0	2.0
n_c (fm $^{-3}$)	0.0025	0.672
p_c (fm $^{-3}$)	0.001	0.269
M/M_\odot	0.009	1.033
R (km)	13.39	10.17
$M(R)/R$	0.001	0.150

TABLE II: Scaled frequencies κ for the r^o - and r^s -modes associated with $m = 2$ and 3.

η	model	$r^o(m=2)$	$r^s(m=2)$	$r^o(m=3)$	$r^s(m=3)$
0	I	0.6663	0.6660	0.4996	0.4995
	II	0.6094	0.5459	0.4411	0.4194
0.04	I	0.6663	0.7592	0.4996	0.5694
	II	0.6094	0.6178	0.4411	0.4754
0.1	I	0.6663	0.8990	0.4996	0.6743
	II	0.6094	0.7240	0.4411	0.5585
0.2	I	0.6663	1.132	0.4996	0.8491
	II	0.6094	0.8964	0.4411	0.6942

FIG. 1: κ 's for r^o - and r^s -modes associated with $m = 2$ in the two models I and II, given as a function of η .

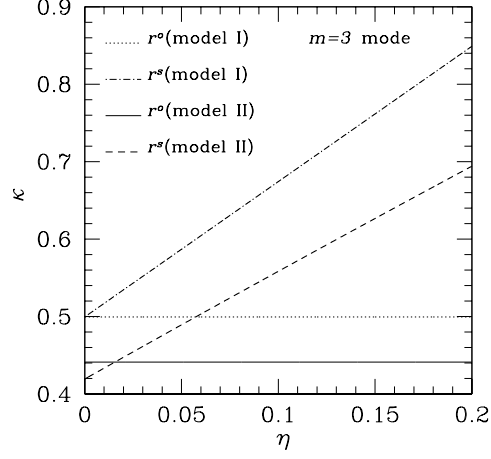


FIG. 2: Same as Figure 1 but for modes associated with $m = 3$.

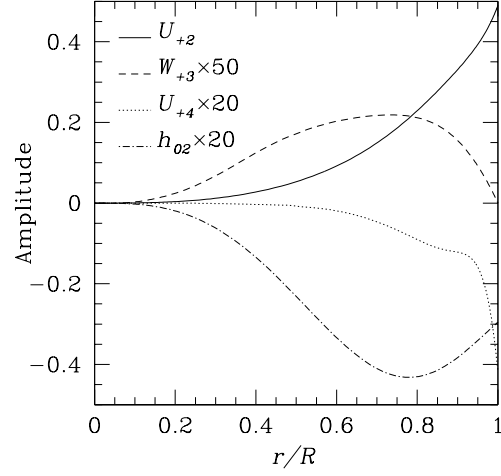


FIG. 3: Four dominant expansion coefficients U_{+2} , W_{+3} , U_{+4} , and h_{02} for the r^o -mode associated with $m = 2$ in model II for $\eta = 0.1$, given as a function of r/R . Other coefficients have negligible amplitude. The corresponding κ is given by $\kappa = 0.6094$. The amplitudes are normalized by $U_{n2} = 1$ at $r = R$.

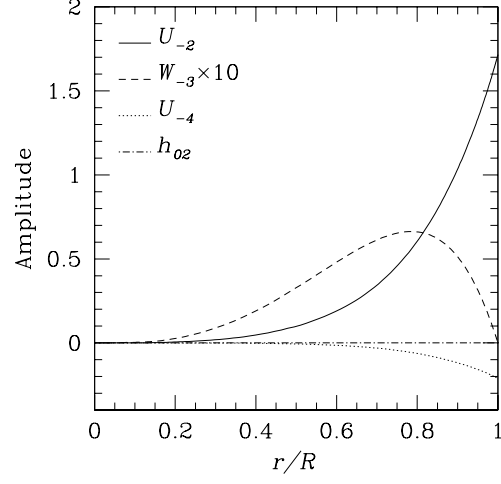


FIG. 4: Four dominant expansion coefficients U_{-2} , W_{-3} , U_{-4} , and h_{02} for the r^s -mode associated with $m = 2$ in model II for $\eta = 0.1$, given as a function of r/R . Other coefficients have negligible amplitude. The corresponding κ is given by $\kappa = 0.7240$. The amplitudes are normalized by $U_{n2} = 1$ at $r = R$.